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Basic Sturm–Liouville problems

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Abstract
This paper is devoted to studying a q-analogue of Sturm–Liouville eigenvalue problems. We formulate a self-adjoint q-difference operator in a Hilbert space. Some of the properties of the eigenvalues and the eigenfunctions are discussed. Green's function is constructed and the problem in question is inverted into a q-type Fredholm integral operator with a symmetric kernel. The set of eigenfunctions is shown to be a complete orthogonal set in the Hilbert space. Examples involving basic trigonometric functions are involved.

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1. Introduction

Let $[a, b] \subseteq \mathbb{R}$ be a finite closed interval and $\nu(\cdot)$ be a continuous real-valued function defined on $[a, b]$. By a Sturm–Liouville problem we mean the problem of finding a function $y(\cdot)$ and a number $\lambda \in \mathbb{C}$ satisfying the differential equation

$$Ly := -y'' + \nu(x)y(x) = \lambda y(x), \qquad a \leqslant x \leqslant b, \tag{1.1}$$

together with the boundary conditions

$$U_1(y) := a_1 y(a) + a_2 y'(a) = 0, \tag{1.2}$$

$$U_2(y) := b_1 y(b) + b_2 y'(b) = 0, \tag{1.3}$$

where a_i and b_i, $i = 1, 2$, are real numbers for which

$$|a_1| + |a_2| \neq 0 \neq |b_1| + |b_2|. \tag{1.4}$$

This problem has been extensively studied. It is known that the differential equation (1.1) and the boundary conditions (1.2), (1.3) determine a self-adjoint operator in $L^2(a, b)$. There is a sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ with ∞ as the unique limit point such that corresponding to each λ_n there is one and only one linearly independent solution of the problem (1.1)–(1.3).

The sequence $\{\lambda_n\}_{n=0}^{\infty}$ is called the sequence of eigenvalues and the sequence of corresponding solutions $\{\phi_n(\cdot)\}_{n=0}^{\infty}$ is said to be a sequence of eigenfunctions. One of the most important properties of these eigenfunctions is that $\{\phi_n(\cdot)\}_{n=0}^{\infty}$ is an orthogonal basis of $L^2(a,b)$. For example, let $\nu(x) \equiv 0$ on $[a,b]$. If we take $a = 0$, $b = \pi$, $a_1 = 1$, $a_2 = 0$, $b_1 = 1$ and $b_2 = 0$, we get

$$\lambda_n = n^2, \qquad \phi_n(x) = \sin nx, \qquad n = 1, 2, \ldots, \tag{1.5}$$

leading to the well-known fact that $\{\sin nx\}_{n=1}^{\infty}$ is a complete orthogonal set of $L^2(0,\pi)$, while taking $a = 0$, $b = \pi$, $a_1 = 0$, $a_2 = 1$, $b_1 = 0$ and $b_2 = 1$, we get

$$\lambda_n = n^2, \qquad \phi_n(x) = \cos nx, \qquad n = 0, 1, 2, \ldots, \tag{1.6}$$

which leads to the completeness of $\{\cos nx\}_{n=0}^{\infty}$ in $L^2(0,\pi)$. We mention here that the Fourier orthogonal basis $\{\mathrm{e}^{\mathrm{i}nx}\}_{n=0}^{\infty}$ of $L^2(-\pi,\pi)$ will not be extracted from this setting but from a simpler situation, namely the first-order problem

$$-\mathrm{i}y' = \lambda y, \qquad y(-\pi) = y(\pi). \tag{1.7}$$

Among several references for the above-mentioned facts we mention the monographs of Coddington and Levinson [12], Estham [13], Levitan and Saragsjan [30, 31], Marchenko [33] and Titchmarsh [39].

The discrete analogue of the theory outlined above, i.e. when the differential operator $\mathrm{d}/\mathrm{d}x$ is replaced by the forward difference operator $\Delta y(n) = y(n+1) - y(n)$ and the backward operator $\nabla y(n) = y(n) - y(n-1)$ where n is a positive integer belonging to a finite set of integers of the form $\{m, m+1, m+2, \ldots, m+N, m \geqslant 1\}$, is treated in Atkinson's [5] (see also [27]).

The aim of this paper is to study a basic analogue of Sturm–Liouville systems when the differential operator is replaced by the q-difference operator D_q (see (2.12), (2.13)). In [14, chapter 5] and [15], a basic Sturm–Liouville system is defined. It is the system

$$D_q(r(x)D_q y) + (l(x) + \lambda w(x))y(qx) = 0, \qquad a \leqslant x \leqslant b, \tag{1.8}$$

$$h_1 y(a) + h_2 D_q y(a) = 0, \tag{1.9}$$

$$k_1 y(b) + k_2 D_q y(b) = 0, \tag{1.10}$$

where $r(\cdot)$, $l(\cdot)$ and $w(\cdot)$ are the real-valued functions which possess appropriate q-derivatives, h_1, h_2, k_1, k_2 are constants. It is proved [14, pp 164–70] that all eigenvalues of this system are real and the eigenfunctions satisfy an orthogonality relation [14, equation (5.1.5)]. Exton [14] considered only formal computational aspects of this problem in order to prove certain orthogonality relations of some q-special functions. There is no attention paid to several points, which may lead to several mistakes. First the existence of eigenvalues is not proved and it is not indicated how to determine the eigenvalues and the eigenfunctions. A basic point here is that if $a \neq 0 \neq b$, then it is not guaranteed that initial conditions at either a or b determine a unique solution of (1.8) (see [32]). The geometric and algebraic simplicity of the eigenvalues, which play a major role in proving the reality of the eigenvalues and the orthogonality of the eigenfunctions, are not proved or even assumed. Moreover the space where the problem is defined is not specified. If an inner product is defined in the view of [14, equation (5.1.5)], there will be no orthogonality if h_1, h_2, k_1 and k_2 are not real. For more information concerning the monograph [14], see the review by Ismail in [21]. See also the review of [15] by Hahn in [20]. In the present paper we formulate a self-adjoint basic Sturm–Liouville eigenvalue problem in a Hilbert space. We prove the existence of a sequence of real eigenvalues with no finite limit points. The associated Green function is constructed and the equivalence between the basic Sturm–Liouville problem and a q-type Fredholm integral operator is proved. Illustrative examples are given at the end of this paper.

There are several physical models involving q-(basic) derivatives, q-integrals, q-functions and their related problems (see, e.g., [11, 16, 17, 19, 34]). Also the problem of expendability of functions in terms of q-orthogonal functions, which seems to be first discussed by Carmichael in [9, 10], has attracted the work of several authors (see, e.g., [7, 8, 23, 35, 36]). However, as far as we know, there is no study of the general problem as we do in the present setting. At this point, it is worth mentioning that our work based on the q-difference operator which is attributed to Jackson, see [25], and a similar study of the Stum–Liouville systems generated by the Askey–Wilson derivative, cf [4] is very much needed.

2. q-Notation and results

In this section we introduce some of the required q-notation and results. Throughout this paper q is a positive number with $0 < q < 1$. We start with the q-shifted factorial, see [18], for $a \in \mathbb{C}$,

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1}(1 - aq^i), & n = 1, 2, \ldots. \end{cases} \tag{2.1}$$

The multiple q-shifted factorial for complex numbers $a_1, \ldots, a_k$ is defined by

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j;q)_n. \tag{2.2}$$

The limit of $(a;q)_n$ as n tends to infinity exists and will be denoted by $(a;q)_\infty$. Let ${}_r\phi_s$ denote the q-hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle|\, q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} x^n (-q^{(n-1)/2})^{n(s+1-r)}. \tag{2.3}$$

The third type of q-Bessel function is defined by

$$J_\nu^{(3)}(x;q) := x^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}\middle|\, q; qx^2\right), \qquad \nu > -1, \tag{2.4}$$

see [22, 24]. This function is called in some literature the Hahn–Exton q-Bessel function (see [37]). Since the other types of the q-Bessel functions, i.e. $J_\nu^{(1)}(\cdot;q)$, $J_\nu^{(2)}(\cdot;q)$, see [22], will not be used here we use the notation $J_\nu(\cdot;q)$ for $J_\nu^{(3)}(\cdot;q)$. The basic trigonometric functions $\cos(x;q)$ and $\sin(x;q)$ are defined on $\mathbb{C}$ by

$$\cos(x;q) := \sum_{n=0}^{\infty}(-1)^n \frac{q^{n^2}(x(1-q))^{2n}}{(q;q)_{2n}} \tag{2.5}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(x(1-q))^{1/2} J_{-1/2}(x(1-q)/\sqrt{q};q^2), \tag{2.6}$$

$$\sin(x;q) := \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}(x(1-q))^{2n+1}}{(q;q)_{2n+1}} \tag{2.7}$$

$$= \frac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty}(x(1-q))^{1/2} J_{1/2}(x(1-q);q^2), \tag{2.8}$$

and they are q-analogues of the cosine and sine functions [2, 18]. It is proved in [29] that $J_\nu(\cdot, q^2)$ has an infinite number of real and simple zeros only. Moreover, cf [1], if $w_m^{(\nu)}$, $m \geqslant 1$, denote the positive zeros of $J_\nu(\cdot, q^2)$ and $q^{2\nu+2} < (1-q^2)^2$, then

$$w_m^{(\nu)} = q^{-m+\epsilon_m^{(\nu)}}, \qquad \sum_{m=1}^{\infty} \epsilon_m^{(\nu)} < \infty, \qquad 0 \leqslant \epsilon_m^{(\nu)} < 1. \tag{2.9}$$

It is also proved in [1] that for any $q \in (0, 1)$, the zeros $\{w_m^{(\nu)}\}_{m=0}^{\infty}$ of $J_\nu(z; q^2)$ have form (2.9) for sufficiently large m. From (2.6), (2.8) and (2.9) $\sin(\cdot; q)$ and $\cos(\cdot; q)$ have only real and simple zeros $\{0, \pm x_m\}_{m=1}^{\infty}$ and $\{\pm y_m\}_{m=1}^{\infty}$ respectively, where $x_m, y_m > 0$, $m \geqslant 1$ and

$$x_m = (1-q)^{-1} q^{-m+\epsilon_m^{(-1/2)}}, \qquad \text{if} \quad q < (1-q^2)^2, \tag{2.10}$$

$$y_m = (1-q)^{-1} q^{-m+1/2+\epsilon_m^{(1/2)}}, \qquad \text{if} \quad q^3 < (1-q^2)^2. \tag{2.11}$$

Moreover, for any $q \in (0, 1)$, (2.10) and (2.11) hold for sufficiently large m.

For $\mu \in \mathbb{R}$, a set $A \subseteq \mathbb{R}$ is called a μ-geometric set if $\mu x \in A$ for all $x \in A$. If $A \subseteq \mathbb{R}$ is a μ-geometric set, then it contains all geometric sequences $\{x\mu^n\}_{n=0}^{\infty}$, $x \in A$. Let f be a function, real or complex-valued, defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \qquad x \in A/\{0\}. \tag{2.12}$$

If $0 \in A$, the q-derivative at zero is defined by

$$D_q f(0) := \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \qquad x \in A, \tag{2.13}$$

if the limit exists and does not depend on x. A right inverse to D_q, the Jackson q-integration, cf [26], is given by

$$\int_0^x f(t)\,\mathrm{d}_q t := x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n), \qquad x \in A, \tag{2.14}$$

provided that the series converges, and

$$\int_a^b f(t)\,\mathrm{d}_q t := \int_0^b f(t)\,\mathrm{d}_q t - \int_0^a f(t)\,\mathrm{d}_q t, \qquad a, b \in A. \tag{2.15}$$

Lemma 2.1. *The necessary and sufficient condition for the existence of the q-integral (2.14) is that* $\lim_{k\to\infty} xq^k f(xq^k) = 0$.

Proof. Based on the fact that, for $x \in \mathbb{C}$

$$\lim_{k\to\infty} xq^k f(xq^k) = 0 \iff \exists \alpha \in [0, 1[\exists C > 0, |f(xq^k)| \leqslant C|xq^k|^{-\alpha}, \qquad k \in \mathbb{N}. \quad \square$$

Kac and Cheung [28, p 68] have proved that if $x^\alpha f(x)$ is bounded on $[0, a]$ for some $0 \leqslant \alpha < 1$, then $\int_0^x f(t)\,\mathrm{d}_q t$ exists for all $x \in [0, a]$. It is not hard to see that

$$D_q \int_0^x f(t)\,\mathrm{d}_q t = f(x), \qquad \int_0^a D_q f(t)\,\mathrm{d}_q t = f(a) - f(0). \tag{2.16}$$

The second identity in (2.16) occurs when $f(\cdot)$ is q-regular at zero, i.e.

$$\lim_{n\to\infty} f(xq^n) = f(0), \qquad \text{for all} \quad x \in A. \tag{2.17}$$

The non-symmetric Leibniz' rule

$$D_q(fg)(x) = g(x)D_qf(x) + f(qx)D_qg(x) \tag{2.18}$$

holds. Relation (2.18) can be symmetrized using the relation $f(qx) = f(x) - x(1-q)D_qf(x)$, giving the additional term $x(1-q)D_qf(x)D_qg(x)$. Also the rule of q-integration by parts is nothing but

$$\int_0^a g(x)D_qf(x)\,\mathrm{d}_qx = (fg)(a) - \lim_{n\to\infty}(fg)(aq^n) - \int_0^a D_qg(x)f(qx)\,\mathrm{d}_qx. \tag{2.19}$$

If f, g are q-regular at zero, then $\lim_{n\to\infty}(fg)(aq^n)$ on the right-hand side of (2.19) will be replaced by $(fg)(0)$. A self-contained q-calculus may be found in [32]. In the following we define Hilbert spaces where our q-Sturm–Liouville problem will be considered. Let $L_q^2(0,a)$ be the space of all complex-valued functions defined on $[0,a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|^2\,\mathrm{d}_qx\right)^{1/2} < \infty. \tag{2.20}$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$\langle f, g\rangle := \int_0^a f(x)\overline{g(x)}\,\mathrm{d}_qx, \qquad f, g \in L_q^2(0,a), \tag{2.21}$$

and the orthonormal basis

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\sqrt{x(1-q)}}, & x = aq^n,\\ 0, & \text{otherwise,}\end{cases} \tag{2.22}$$

$n = 0, 1, 2, \ldots,$ cf [3]. The space $L_q^2((0,a)\times(0,a))$ is the space of all complex-valued functions $f(x,t)$ defined on $[0,a]\times[0,a]$ such that

$$\|f(\cdot,\cdot)\|_2 := \left(\int_0^a\int_0^a |f(x,t)|^2\,\mathrm{d}_qx\,\mathrm{d}_qt\right)^{1/2} < \infty. \tag{2.23}$$

Lemma 2.2. $L_q^2((0,a)\times(0,a))$ *is a separable Hilbert space with the inner product*

$$\langle f, g\rangle_2 := \int_0^a\int_0^a f(x,t)\overline{g(x,t)}\,\mathrm{d}_qx\,\mathrm{d}_qt. \tag{2.24}$$

Proof. Similar to [3, pp 217–8], $L_q^2((0,a)\times(0,a))$ is a Banach space. The elements of $L_q^2((0,a)\times(0,a))$ are equivalence classes where f, g are in the same equivalence class if $f(aq^m, aq^n) = g(aq^m, aq^n), m, n \in \mathbb{N}$. The zero element is the equivalence class of all functions $f(x,t)$ which satisfy $f(aq^m, aq^n) = 0$, for all $m, n \in \mathbb{N}$. To prove separability, it suffices to prove that

$$\phi_{ij}(x,t) := \phi_i(x)\phi_j(t), \quad i, j = 1, 2, \ldots, \tag{2.25}$$

is an orthonormal basis of $L_q^2((0,a)\times(0,a))$ whenever $\{\phi_i(\cdot)\}_{i=1}^\infty$ is an orthonormal basis of $L_q^2(0,a)$. Indeed,

$$\begin{aligned}\langle\phi_{jk}, \phi_{mn}\rangle_2 &= \int_0^a\int_0^a \phi_j(x)\phi_k(t)\overline{\phi_m}(x)\overline{\phi_n}(t)\,\mathrm{d}_qx\,\mathrm{d}_qt\\ &= \int_0^a \phi_j(x)\overline{\phi_m}(x)\,\mathrm{d}_qx\int_0^a \phi_k(t)\overline{\phi_n}(t)\,\mathrm{d}_qt = \delta_{jm}\delta_{kn},\end{aligned}$$

proving orthogonality. To prove that $\{\phi_{ij}\}$ is a basis, we prove that if there exists $f \in L_q^2((0,a)\times(0,a))$ such that $\langle f, \phi_{ij}\rangle_2 = 0$, then f is the zero element. Indeed,

$$0 = \langle f, \phi_{ij}\rangle = \int_0^a \int_0^a f(x,t)\overline{\phi_i(x)\phi_j(t)}\,\mathrm{d}_q x\,\mathrm{d}_q t$$
$$= \int_0^a \overline{\phi_j(t)} \left(\int_0^a f(x,t)\overline{\phi_i(x)}\,\mathrm{d}_q x \right) \mathrm{d}_q t = \int_0^a h(t)\overline{\phi_j(t)}\,\mathrm{d}_q t.$$

Thus

$$h(t) := \int_0^a f(x,t)\overline{\phi_i(x)}\,\mathrm{d}_q x \tag{2.26}$$

is orthogonal to the ϕ_j which implies that $h(aq^n) = 0$ for all $n \in \mathbb{N}$. So, $f(x, aq^n)$ is orthogonal to each ϕ_i. Consequently, $f(aq^m, aq^n) = 0$, for all $m, n \in \mathbb{N}$. □

3. Fundamental solutions

In this section we investigate the fundamental solutions of the basic Sturm–Liouville equation

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + \nu(x) y(x) = \lambda y(x), \qquad 0 \leqslant x \leqslant a < \infty, \qquad \lambda \in \mathbb{C}, \tag{3.1}$$

where $\nu(\cdot)$ is defined on $[0,a]$ and continuous at zero. Let $C_q^2(0,a)$ be the space of all functions $y(\cdot)$ defined on $[0,a]$ such that $y(\cdot)$, $D_q y(\cdot)$ are continuous at zero. Clearly, $C_q^2(0,a)$ is a subspace of the Hilbert space $L_q^2(0,a)$. By a solution of equation (3.1), we mean a continuous at zero function that satisfies (3.1) such that the function and its q-derivative have prescribed values at $x = 0$. It is proved in [32] that (3.1) has a fundamental set of solutions which consists of two linearly independent solutions $\{y_1(\cdot), y_2(\cdot)\}$. The q-Wronskian of $y_1(\cdot)$, $y_2(\cdot)$ is defined to be

$$W_q(y_1, y_2)(x) := y_1(x) D_q y_2(x) - y_2(x) D_q y_1(x), \qquad x \in [0,a]. \tag{3.2}$$

$\{y_1, y_2\}$ forms a fundamental set of solutions if and only if their q-Wronskian does not vanish at any point of $[0,a]$. See [32, 38].

Theorem 3.1. *For $c_1, c_2 \in \mathbb{C}$, equation (3.1) has a unique solution in $C_q^2(0,a)$ which satisfies*

$$\phi(0,\lambda) = c_1, \qquad D_{q^{-1}}\phi(0,\lambda) = c_2, \qquad \lambda \in \mathbb{C}. \tag{3.3}$$

Moreover $\phi(x,\lambda)$ is entire in λ for all $x \in [0,a]$, where the q^{-1}-derivative of a function $f(x)$ at zero is given by

$$D_{q^{-1}} f(x) = \lim_{n\to-\infty} \frac{f(xq^{-n}) - f(0)}{xq^{-n}} = D_q f(0). \tag{3.4}$$

Proof. The functions

$$\varphi_1(x,\lambda) = \cos(sx;q), \qquad \text{and} \qquad \varphi_2(x,\lambda) = \begin{cases} \dfrac{\sin(sx;q)}{s}, & \lambda \neq 0 \\ x, & \lambda = 0, \end{cases} \tag{3.5}$$

where $s := \sqrt{\lambda}$ is defined with respect to the principal branch, are a fundamental set of

$$\frac{1}{q} D_{q^{-1}} D_q y(x) + \lambda y(x) = 0, \tag{3.6}$$

with the q-Wronskian $W_q(\varphi_1(\cdot,\lambda),\varphi_2(\cdot,\lambda)) \equiv 1$ (cf [32]). For all $x \in [0,a]$, $\lambda \in \mathbb{C}$, we define a sequence $\{y_m(\cdot,\lambda)\}_{m=1}^{\infty}$ of successive approximations by

$$y_1(x,\lambda) = c_1\varphi_1(x,\lambda) + c_2\varphi_2(x,\lambda), \tag{3.7}$$

$$\begin{aligned} y_{m+1}(x,\lambda) &= c_1\varphi_1(x,\lambda) + c_2\varphi_2(x,\lambda) \\ &\quad + q\int_0^x \{\varphi_2(x,\lambda)\varphi_1(qt,\lambda) - \varphi_1(x,\lambda)\varphi_2(qt,\lambda)\}\nu(qt)y_m(qt,\lambda)\,\mathrm{d}_q t. \end{aligned} \tag{3.8}$$

We prove that for each fixed $\lambda \in \mathbb{C}$ the uniform limit of $\{y_m(\cdot,\lambda)\}_{m=1}^{\infty}$ as $m \to \infty$ exists and defines a solution of (3.1) and (3.3). Let $\lambda \in \mathbb{C}$ be fixed. There exist positive numbers $K(\lambda)$ and A such that

$$|\nu(x)| \leqslant A, \qquad |\varphi_i(x,\lambda)| \leqslant \sqrt{\frac{K(\lambda)}{2}}; \qquad i = 1,2; \quad x \in [0,a]. \tag{3.9}$$

Let $\widetilde{K}(\lambda) := (|c_1|+|c_2|)\sqrt{\frac{K(\lambda)}{2}}$. Then, from (3.9), $|y_1(x,\lambda)| \leqslant \widetilde{K}(\lambda)$, for all $x \in [0,a]$. Using mathematical induction, we have

$$|y_{m+1}(x,\lambda) - y_m(x,\lambda)| \leqslant \widetilde{K}(\lambda)q^{\frac{m(m+1)}{2}}\frac{(AK(\lambda)x(1-q))^m}{(q;q)_m}, \qquad m = 1,2,\ldots. \tag{3.10}$$

Consequently by Weierstrass' test the series

$$y_1(x,\lambda) + \sum_{m=1}^{\infty} y_{m+1}(x,\lambda) - y_m(x,\lambda) \tag{3.11}$$

converges uniformly in $[0,a]$. Since the mth partial sum of the series is nothing but $y_{m+1}(\cdot,\lambda)$, then $y_{m+1}(\cdot,\lambda)$ approaches a function $\phi(\cdot,\lambda)$ uniformly in $[0,a]$ as $m \to \infty$, where $\phi(x,\lambda)$ is the sum of the series. We can also prove by induction on m that $y_m(x,\lambda)$ and $D_q y_m(x,\lambda)$ are continuous at zero, where

$$\begin{aligned} D_q y_{m+1}(x,\lambda) &= c_1 D_q\varphi_1(x,\lambda) + c_2 D_q\varphi_2(x,\lambda) \\ &\quad + q\int_0^x \{D_q\phi_2(x,\lambda)\varphi_1(qt,\lambda) - D_q\varphi_1(x,\lambda)\varphi_2(qt,\lambda)\}\nu(qt)y_m(qt,\lambda)\,\mathrm{d}_q t, \end{aligned} \tag{3.12}$$

$m = 1,2,\ldots$. Hence, both $\phi(\cdot,\lambda)$ and $D_q\phi(\cdot,\lambda)$ are continuous at zero, i.e. $\phi(\cdot,\lambda) \in C_q^2(0,a)$. Because of the uniform convergence, letting $m \to \infty$ in (3.8) we obtain

$$\begin{aligned} \phi(x,\lambda) &= c_1\varphi_1(x,\lambda) + c_2\varphi_2(x,\lambda) \\ &\quad + q\int_0^x \{\varphi_2(x,\lambda)\varphi_1(qt,\lambda) - \varphi_1(x,\lambda)\varphi_2(qt,\lambda)\}\nu(qt)\phi(qt,\lambda)\,\mathrm{d}_q t. \end{aligned} \tag{3.13}$$

We prove that $\phi(\cdot,\lambda)$ satisfies (3.1) and (3.3). Clearly $\phi(0,\lambda) = c_1$ and

$$D_{q^{-1}}\phi(0,\lambda) = \lim_{n\to\infty}\frac{\phi(xq^n,\lambda) - \phi(0,\lambda)}{xq^n} = c_1 D_{q^{-1}}\phi_1(0,\lambda) + c_2 D_{q^{-1}}\phi_2(0,\lambda) = c_2, \tag{3.14}$$

i.e. $\phi(x,\lambda)$ satisfies (3.3). To prove that $\phi(\cdot,\lambda)$ satisfies (3.1), we distinguish between two cases, $x \neq 0$ and $x = 0$. If $x \neq 0$, then from (2.16),

$$\begin{aligned} D_q\phi(x,\lambda) &= c_1 D_q\varphi_1(x,\lambda) + c_2 D_q\varphi_2(x,\lambda) \\ &\quad + q\int_0^x \{D_q\phi_2(x,\lambda)\varphi_1(qt,\lambda) - D_q\varphi_1(x,\lambda)\varphi_2(qt,\lambda)\}\nu(qt)\phi(qt,\lambda)\,\mathrm{d}_q t \end{aligned} \tag{3.15}$$

and hence

$$\frac{-1}{q}D_{q^{-1}}D_q\phi(x,\lambda)=\frac{-1}{q}D_{q^{-1}}D_q\varphi_1(x,\lambda)\left(c_1-\int_0^x\varphi_2(qt,\lambda)\nu(qt)\phi(qt,\lambda)\,\mathrm{d}_qt\right)$$
$$-\frac{1}{q}D_{q^{-1}}D_q\varphi_2(x,\lambda)\left(c_2+\int_0^x\varphi_1(qt,\lambda)\nu(qt)\phi(qt,\lambda)\,\mathrm{d}_qt\right)-\nu(x)\phi(x,\lambda). \tag{3.16}$$

Substituting from (3.5), (3.6) and (3.13) into (3.16), we conclude that $\phi(x,\lambda)$ satisfies (3.1) for $x\neq 0$. If $x=0$, then (3.1) is nothing but

$$D_q^2y(0)-q\nu(0)y(0)=-q\lambda y(0). \tag{3.17}$$

Obviously

$$\begin{aligned}D_q^2\phi(0,\lambda)&=c_1D_q^2\varphi_1(0,\lambda)+c_2D_q^2\varphi_2(0,\lambda)+q\nu(0)\phi(0,\lambda)\\&=-c_1q\lambda\varphi_1(0,\lambda)-c_2q\lambda\varphi_2(0,\lambda)+q\nu(0)\phi(0,\lambda)\\&=-q\lambda\phi(0,\lambda)+q\nu(0)\phi(0,\lambda)\end{aligned} \tag{3.18}$$

Hence $\phi(x,\lambda)$ satisfies (3.1). To prove that problem (3.1), (3.3) has a unique solution, suppose on the contrary that $\psi_i(\cdot,\lambda), i=1,2$, are two solutions of (3.1), (3.3). Let $\chi(x,\lambda)=\psi_1(x,\lambda)-\psi_2(x,\lambda), x\in[0,a]$. Then $\chi(\cdot,\lambda)$ is a solution of (3.1) subject to the initial conditions $\chi(0,\lambda)=D_{q^{-1}}\chi(0,\lambda)=0$. Applying the q-integration process twice to (3.1) yields

$$\chi(x,\lambda)=\int_0^x(x-t)(\lambda-\nu(t))\chi(t,\lambda)\,\mathrm{d}_qt. \tag{3.19}$$

Since $\chi(x,\lambda),\nu(x)$ are continuous at zero, then there exist positive numbers $N_{x,\lambda}, M_{x,\lambda}$ such that

$$N_{x,\lambda}=\sup_{n\in\mathbb{N}}|\chi(xq^n,\lambda)|,\qquad M_{x,\lambda}=\sup_{n\in\mathbb{N}}|\lambda-\nu(xq^n)|. \tag{3.20}$$

Again we can prove by mathematical induction on k that

$$|\chi(x,\lambda)|\leqslant N_{x,\lambda}M_{x,\lambda}^kq^{k^2}(1-q)^{2k}\frac{x^{2k}}{(q;q)_{2k}},\qquad k\in\mathbb{N},\quad x\in[0,a]. \tag{3.21}$$

Indeed, if (3.21) holds at $k\in\mathbb{N}$, then from (3.19)

$$\begin{aligned}|\chi(x,\lambda)|&\leqslant N_{x,\lambda}M_{x,\lambda}^{k+1}q^{k^2}\frac{(1-q)^{2k}}{(q;q)_{2k}}\int_0^x(x-t)t^{2k}\,\mathrm{d}_qt\\&=N_{x,\lambda}M_{x,\lambda}^{k+1}q^{k^2}q^{2k+1}\frac{(1-q)^{2k+2}}{(q;q)_{2k+2}}x^{2k+2}\\&=N_{x,\lambda}M_{x,\lambda}^{k+1}q^{(k+1)^2}\frac{(1-q)^{2k+2}}{(q;q)_{2k+2}}x^{2k+2}.\end{aligned} \tag{3.22}$$

Hence (3.21) holds at $k+1$. Consequently (3.21) holds for all $k\in\mathbb{N}$ because from (3.20) it holds at $k=0$. Since $\lim_{k\to\infty}M_{x,\lambda}^kq^{k^2}(1-q)^{2k}\frac{x^{2k}}{(q;q)_{2k}}=0$, then $\chi(x,\lambda)=0$, for all $x\in[0,a]$. This proves the uniqueness. Now, Let $M>0$ be arbitrary but fixed. To prove that $\phi(x,\lambda), x\in[0,a]$, is entire in λ, it is sufficient to prove that $\phi(x,\lambda)$ is analytic in each disc Ω_M; $\Omega_M:=\{\lambda\in\mathbb{C}:|\lambda|\leqslant M\}$. We prove by induction on m that

$$\text{for all}\quad x\in[0,a]\quad y_m(x,\lambda)\quad\text{is analytic on }\Omega_M, \tag{3.23}$$

$$\text{for all}\quad \lambda\in\Omega_M\quad \frac{\partial}{\partial\lambda}y_m(x,\lambda)\quad\text{is continuous at }(0,\lambda). \tag{3.24}$$

Clearly, $\varphi_1(x,\lambda), \varphi_2(x,\lambda)$ are entire functions of λ for $x \in [0,a]$. Moreover $\frac{\partial}{\partial\lambda}\phi_i(x,\lambda)$ is continuous at $(0,\lambda)$ for each $\lambda \in \mathbb{C}$. Then (3.23) and (3.24) hold at $m = 1$. Assume that (3.23) and (3.24) hold at $m \geqslant 1$. Then for $x_0 \in [0,a]$, $\lambda_0 \in \Omega_M$, we obtain

$$\begin{aligned}\left.\frac{\partial y_{m+1}(x_0,\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0} &= q\left.\frac{\partial\varphi_2(x_0,\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0}\int_0^{x_0}\varphi_1(qt,\lambda)y_m(qt,\lambda)\,\mathrm{d}_qt\\ &\quad+\left.\frac{\partial y_1(x_0,\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0}-q\left.\frac{\partial\varphi_1(x_0,\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0}\int_0^{x_0}\varphi_2(qt,\lambda)y_m(qt,\lambda)\,\mathrm{d}_qt\\ &\quad+q\varphi_2(x_0,\lambda)\frac{\partial}{\partial\lambda}\left.\left(\int_0^{x_0}\varphi_1(qt,\lambda)y_m(qt,\lambda)\,\mathrm{d}_qt\right)\right|_{\lambda=\lambda_0}\\ &\quad-q\varphi_1(x_0,\lambda)\frac{\partial}{\partial\lambda}\left.\left(\int_0^{x_0}\varphi_2(qt,\lambda)y_m(qt,\lambda)\,\mathrm{d}_qt\right)\right|_{\lambda=\lambda_0}.\end{aligned}\tag{3.25}$$

From (3.24) we conclude that $\partial/\partial\lambda(\varphi_i(qt,\lambda)y_m(qt,\lambda))$, $i = 1,2$, are continuous at $(0,\lambda_0)$. Therefore there exist constants $C, \delta > 0$ such that

$$\left|\frac{\partial}{\partial\lambda}(\varphi_i(x_0q^n,\lambda)y_m(x_0q^n,\lambda))\right| \leqslant C, \qquad n\in\mathbb{N}, \quad |\lambda-\lambda_0|\leqslant\delta. \tag{3.26}$$

Hence

$$x_0(1-q)q^n\left|\frac{\partial}{\partial\lambda}(\varphi_i(x_0q^{n+1},\lambda)y_m(x_0q^{n+1}\lambda))\right| \leqslant x_0A(1-q)q^n, \qquad n\in\mathbb{N},$$

for all λ in the disc $|\lambda-\lambda_0|\leqslant\delta$. i.e. the series corresponding to the q-integrals

$$\int_0^{x_0}\frac{\partial}{\partial\lambda}(\varphi_i(qt,\lambda)y_m(qt,\lambda))\,\mathrm{d}_qt, \qquad i = 1,2 \tag{3.27}$$

are uniformly convergent in a neighbourhood of $\lambda = \lambda_0$. Thus, we can interchange the differentiation and integration processes in (3.25). Since x_0, λ_0 are arbitrary, then

$$\begin{aligned}\frac{\partial}{\partial\lambda}y_{m+1}(x,\lambda) &= \frac{\partial}{\partial\lambda}y_1(x,\lambda) - q\int_0^x\frac{\partial}{\partial\lambda}(\varphi_2(x,\lambda)\varphi_1(qt,\lambda)y_m(qt,\lambda))v(qt)\,\mathrm{d}_qt\\ &\quad+q\int_0^x\frac{\partial}{\partial\lambda}(\varphi_1(x,\lambda)\varphi_2(qt,\lambda)y_m(qt,\lambda))v(qt)\,\mathrm{d}_qt,\end{aligned}\tag{3.28}$$

for all $x\in[0,a], \lambda\in\Omega_M$. From (3.24) the integrals in (3.28) are continuous at $(0,\lambda)$. Consequently $\frac{\partial}{\partial\lambda}y_{m+1}(x,\lambda)$ is continuous at $(0,\lambda)$. Let $x_0\in[0,a]$ be arbitrary. Then there exists $B(x_0), \widetilde{B}(x_0) > 0$ such that

$$|\varphi_i(x_0,\lambda)| \leqslant \sqrt{\frac{B(x_0)}{2}}, \qquad i=1,2, \qquad |y_1(x,\lambda)|\leqslant\widetilde{B}(x_0), \qquad \lambda\in\Omega_M. \tag{3.29}$$

Finally the use of the mathematical induction yields

$$|y_{m+1}(x_0,\lambda)-y_m(x_0,\lambda)| \leqslant \widetilde{B}(x_0)q^{\frac{m(m+1)}{2}}\frac{(AB(x_0)\lambda(1-q))^m}{(q;q)_m}. \tag{3.30}$$

Consequently the series (3.11), with $x = x_0$, converges uniformly in Ω_M to $\phi(x_0,\lambda)$. Hence $\phi(x_0,\lambda)$ is analytic in Ω_M, i.e. it is entire. □

4. The self-adjoint problem

In this section we define a basic Sturm–Liouville problem and prove that it is self-adjoint in $L_q^2(0,a)$. The following lemma which is needed in the following indicates that unlike the classical differential operator d/dx, D_q is neither self-adjoint nor skew self-adjoint. Equation (4.2) indicates that the adjoint of D_q is $-\frac{1}{q}D_{q^{-1}}$.

Lemma 4.1. *Let $f(\cdot)$, $g(\cdot)$ in $L_q^2(0,a)$ be defined on $[0, q^{-1}a]$. Then, for $x \in (0,a]$, we have*

$$(D_q g)(xq^{-1}) = D_{q,xq^{-1}}g(xq^{-1}) = D_{q^{-1}}g(x), \tag{4.1}$$

$$\langle D_q f, g\rangle = f(a)\overline{g(aq^{-1})} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^{n-1})} + \left\langle f, \frac{-1}{q}D_{q^{-1}}g\right\rangle, \tag{4.2}$$

$$\left\langle -\frac{1}{q}D_{q^{-1}}f, g\right\rangle = \lim_{n\to\infty} f(aq^{n-1})\overline{g(aq^n)} - f(aq^{-1})\overline{g(a)} + \langle f, D_q g\rangle. \tag{4.3}$$

Proof. Relation (4.1) follows from

$$D_{q^{-1}}g(x) = \frac{g(x) - g(q^{-1}x)}{x(1-q^{-1})} = \frac{g(xq^{-1}) - g(x)}{xq^{-1}(1-q)} = (D_q g)(xq^{-1}) = D_{q,xq^{-1}}g(xq^{-1}). \tag{4.4}$$

Using formula (2.19) of q-integration by parts we obtain

$$\begin{aligned}
\langle D_q f, g\rangle &= \int_0^a D_q f(x)\overline{g(x)}\,\mathrm{d}_q x \\
&= f(a)\overline{g(a)} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^n)} - \int_0^a f(qt)\overline{D_q g(t)}\,\mathrm{d}_q t \\
&= f(a)\overline{g(a)} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^n)} - \int_0^{qa} f(t)\frac{1}{q}\overline{D_{q^{-1}}g(t)}\,\mathrm{d}_q t \\
&= f(a)\overline{g(a)} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^n)} + aq^{-1}(1-q)f(a)\overline{D_{q^{-1}}g(a)} \\
&\quad + \int_0^a f(t)\overline{\frac{-1}{q}D_{q^{-1}}g(t)}\,\mathrm{d}_q t \\
&= f(a)\overline{g(aq^{-1})} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^{n-1})} + \left\langle f, \frac{-1}{q}D_q g\right\rangle,
\end{aligned} \tag{4.5}$$

proving (4.2). Equation (4.3) can be proved by the use of (4.2). □

Now consider the basic Sturm–Liouville problem

$$\ell(y) := -\frac{1}{q}D_{q^{-1}}D_q y(x) + \nu(x)y(x) = \lambda y(x), \qquad 0 \leqslant x \leqslant a < \infty, \qquad \lambda \in \mathbb{C}, \tag{4.6a}$$

$$U_1(y) := a_{11}y(0) + a_{12}D_{q^{-1}}y(0) = 0, \tag{4.6b}$$

$$U_2(y) := a_{21}y(a) + a_{22}D_{q^{-1}}y(a) = 0, \tag{4.6c}$$

where $\nu(\cdot)$ is a continuous at zero real-valued function and $\{a_{ij}\}$, $i, j \in \{1,2\}$ are arbitrary real numbers such that the rank of the matrix $(a_{ij})_{1\leqslant i,j\leqslant 2}$ is 2. Problem (4.6*a*)–(4.6*c*) is said to be formally self-adjoint if for any functions $y(\cdot)$ and $z(\cdot)$ of $C_q^2(0,a)$ which satisfy (4.6*b*), (4.6*c*),

$$\langle \ell y, z\rangle = \langle y, \ell z\rangle. \tag{4.7}$$

Theorem 4.2. *The basic Sturm–Liouville eigenvalue problem (4.6a)–(4.6c) is formally self-adjoint.*

Proof. We first prove that for $y(\cdot), z(\cdot)$ in $L_q^2(0,a)$, we have the following q-Lagrange's identity

$$\int_0^a (\ell y(x)\overline{z(x)} - y(x)\overline{\ell z(x)})\, \mathrm{d}_q x = [y,z](a) - \lim_{n\to\infty}[y,z](aq^n), \tag{4.8}$$

where

$$[y,z](x) := y(x)\overline{D_{q^{-1}}z(x)} - D_{q^{-1}}y(x)\overline{z(x)}. \tag{4.9}$$

Applying (4.3) with $f(x) = D_q y(x)$ and $g(x) = z(x)$, we obtain

$$\begin{aligned}\left\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\right\rangle &= -(D_q y)(aq^{-1})\overline{z(a)} + \lim_{n\to\infty}(D_q y)(aq^{n-1})\overline{z(aq^n)} + \langle D_q y, D_q z\rangle \\ &= -D_{q^{-1}}y(a)\overline{z(a)} + \lim_{n\to\infty} D_{q^{-1}}y(aq^n)\overline{z(aq^n)} + \langle D_q y, D_q z\rangle.\end{aligned} \tag{4.10}$$

Applying (4.2) with $f(x) = y(x)$, $g(x) = D_q z(x)$,

$$\begin{aligned}\langle D_q y, D_q z\rangle &= y(a)\overline{D_q z(aq^{-1})} - \lim_{n\to\infty} y(aq^n)\overline{D_q z(aq^{n-1})} + \left\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\right\rangle \\ &= y(a)\overline{D_{q^{-1}}z(a)} - \lim_{n\to\infty} y(aq^n)\overline{D_{q^{-1}}z(aq^n)} + \left\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\right\rangle.\end{aligned} \tag{4.11}$$

Therefore,

$$\left\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\right\rangle = [y,z](a) - \lim_{n\to\infty}[y,z](aq^n) + \left\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\right\rangle. \tag{4.12}$$

Lagrange's identity (4.8) results from (4.12) and the reality of $\nu(x)$. Letting $y(\cdot), z(\cdot)$ be in $C_q^2(0,a)$ and assuming that they satisfy (4.6b), (4.6c), we obtain

$$a_{11}y(0) + a_{12}D_{q^{-1}}y(0) = 0, \qquad a_{11}z(0) + a_{12}D_{q^{-1}}z(0) = 0. \tag{4.13}$$

The continuity of $y(\cdot), z(\cdot)$ at zero implies that $\lim_{n\to\infty}[y,z](aq^n) = [y,z](0)$. Then (4.12) will be

$$\left\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\right\rangle = [y,z](a) - [y,z](0) + \left\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\right\rangle. \tag{4.14}$$

Since a_{11} and a_{12} are not both zero, it follows from (4.13) that

$$[y,z](0) = y(0)\overline{D_{q^{-1}}z(0)} - D_{q^{-1}}y(0)\overline{z(0)} = 0.$$

Similarly,

$$[y,z](a) = y(a)\overline{D_{q^{-1}}z(a)} - D_{q^{-1}}y(a)\overline{z(a)} = 0.$$

Since $\nu(x)$ is real valued, then

$$\begin{aligned}\langle \ell(y), z\rangle &= \left\langle -\frac{1}{q}D_{q^{-1}}D_q y(x) + \nu(x)y(x), z(x)\right\rangle \\ &= \left\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\right\rangle + \langle \nu(x)y, z(x)\rangle \\ &= \left\langle y, -\frac{1}{q}D_{q^{-1}}D_q z(x)\right\rangle + \langle y, \nu(x)z(x)\rangle = \langle y, \ell(z)\rangle,\end{aligned}$$

i.e. ℓ is a formally self-adjoint operator. □

A complex number λ^* is said to be an eigenvalue of the problem (4.6a)–(4.6c) if there is a non-trivial solution $\phi^*(\cdot)$ which satisfies the problem at this λ^*. In this case we say that $\phi^*(\cdot)$

is an eigenfunction of the basic Sturm–Liouville problem corresponding to the eigenvalue λ^*. The multiplicity of an eigenvalue is defined to be the number of linearly independent solutions corresponding to it. In particular, an eigenvalue is simple if and only if it has only one linearly independent solution.

Lemma 4.3. *The eigenvalues and the eigenfunctions of the boundary value problem (4.6a)–(4.6c) have the following properties:*

(i) The eigenvalues are real.
(ii) Eigenfunctions that belong to different eigenvalues are orthogonal.
(iii) All eigenvalues are simple from the geometric point of view.

Proof. The proof is similar to that of differential equations, cf [13], and hence it is omitted. □

In the following we indicate how to obtain the eigenvalues and the corresponding eigenfunctions. Let $\phi_1(\cdot,\lambda), \phi_2(\cdot,\lambda)$ be the linearly independent solutions of (4.6a) determined by the initial conditions

$$D_q^{j-1}\phi_i(0,\lambda) = \delta_{ij}, \qquad i, j = 1, 2, \qquad \lambda \in \mathbb{C}. \tag{4.15}$$

Thus $\phi_1(\cdot,\lambda)$ is determined by (3.13) by taking $c_1 = 1, c_2 = 0$ and $\phi_2(\cdot,\lambda)$ is determined by taking $c_1 = 0, c_2 = 1$. Then, every solution of (4.6a) is of the form

$$y(x,\lambda) = A_1\phi_1(x,\lambda) + A_2\phi_2(x,\lambda), \tag{4.16}$$

where A_1 and A_2 do not depend on x. A solution $y(\cdot,\lambda)$ of (4.6a) will be an eigenfunction if it satisfies the boundary conditions (4.6b)–(4.6c), i.e. if we can find a non-trivial solution of the linear system

$$A_1U_1(\phi_1) + A_2U_1(\phi_2) = 0, \qquad A_1U_2(\phi_1) + A_2U_2(\phi_2) = 0, \tag{4.17}$$

Hence, $\lambda \in \mathbb{R}$ is an eigenvalue if and only if

$$\Delta(\lambda) = \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix} = 0. \tag{4.18}$$

The function $\Delta(\lambda)$ defined in (4.18) is called the characteristic determinant associated with the basic Sturm–Liouville problem (4.6a)–(4.6c). The zeros of $\Delta(\lambda)$ are exactly the eigenvalues of the problem. Since $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are entire in λ for each fixed $x \in [0, a]$, then $\Delta(\lambda)$ is also entire. Thus the eigenvalues of the basic Sturm–Liouville system (4.6a)–(4.6c) are at most countable with no finite limit points. From lemma 4.3 we know that all eigenvalues are simple from the geometric point of view. We can prove that the eigenvalues are also simple algebraically, i.e. they are simple zeros of $\Delta(\lambda)$. Indeed, let $\theta_1(\cdot,\lambda)$ and $\theta_2(\cdot,\lambda)$ be defined by the relations

$$\begin{aligned} \theta_1(x,\lambda) &:= U_1(\phi_2)\phi_1(x,\lambda) - U_1(\phi_1)\phi_2(x,\lambda), \\ \theta_2(x,\lambda) &:= U_2(\phi_2)\phi_1(x,\lambda) - U_2(\phi_1)\phi_2(x,\lambda), \end{aligned} \tag{4.19}$$

Hence, $\theta_1(\cdot,\lambda), \theta_2(\cdot,\lambda)$ are solutions of (4.6a) such that

$$\theta_1(0,\lambda) = a_{12}, \quad D_{q^{-1}}\theta_1(0,\lambda) = -a_{11}; \qquad \theta_2(a,\lambda) = a_{22}, \quad D_{q^{-1}}\theta_2(a,\lambda) = -a_{21}. \tag{4.20}$$

One can verify that

$$W_q(\theta_1(\cdot,\lambda), \theta_2(\cdot,\lambda))(x,\lambda) = \Delta(\lambda)W_q(\phi_1(\cdot,\lambda), \phi_2(\cdot,\lambda)) = \Delta(\lambda). \tag{4.21}$$

Let λ_0 be an eigenvalue of (4.6a)–(4.6c). Then λ_0 is a real number and therefore $\theta_i(x, \lambda_0)$ can be taken to be real valued, $i = 1, 2$. From (4.21), we conclude that $\theta_1(x, \lambda_0)$, $\theta_2(x, \lambda_0)$ are linearly dependent eigenfunctions. So, there exists a non-zero constant k_0 such that

$$\theta_1(x, \lambda_0) = k_0\theta_2(x, \lambda_0). \tag{4.22}$$

From (4.19) and (4.20)

$$\theta_1(a, \lambda_0) = k_0 a_{22} = k_0\theta_1(a, \lambda), \qquad D_{q^{-1}}\theta_1(a, \lambda_0) = -k_0 a_{21} = k_0 D_{q^{-1}}\theta_1(a, \lambda). \tag{4.23}$$

In the q-Lagrange identity (4.8), taking $y(x) = \theta_1(x, \lambda)$, and $z(x) = \theta_1(x, \lambda_0)$ implies

$$\begin{aligned}(\lambda - \lambda_0)\int_0^a \theta_1(x, \lambda)\theta_1(x, \lambda_0)\, \mathrm{d}_q x &= \theta_1(a, \lambda)D_{q^{-1}}\theta_1(a, \lambda_0) - D_{q^{-1}}\theta_1(a, \lambda)\theta_1(a, \lambda_0)\\ &= k_0(\theta_1(a, \lambda)D_{q^{-1}}\theta_2(a, \lambda) - \theta_2(a, \lambda)D_{q^{-1}}\theta_1(a, \lambda))\\ &= k_0 W_q(\theta_1(\cdot, \lambda), \theta_2(\cdot, \lambda))(q^{-1}a) = k_0\Delta(\lambda).\end{aligned}$$

Since $\Delta(\lambda)$ is entire in λ,

$$\Delta'(\lambda_0) := \lim_{\lambda\to\lambda_0} \frac{\Delta(\lambda)}{\lambda - \lambda_0} = \frac{1}{k_0}\int_0^a \theta_1^2(x, \lambda_0)\, \mathrm{d}_q x \neq 0. \tag{4.24}$$

Therefore λ_0 is a simple zero of $\Delta(\lambda)$.

5. Basic Green's function

The q-type Green's function arises when we seek a solution of the nonhomogeneous equation

$$-\frac{1}{q}D_{q^{-1}}D_q y(x) + \{-\lambda + v(x)\}y(x) = f(x), \qquad x \in [0, a], \quad \lambda \in \mathbb{C}, \tag{5.1}$$

which satisfies the boundary conditions (4.6b), (4.6c), where $f(\cdot) \in L_q^2(0, a)$ is given. First, we note that if λ is not an eigenvalue of the Sturm–Liouville problem (4.6a)–(4.6c), then the solution of (5.1), if it exists, would be unique. To see this, assume that $\chi_1(x, \lambda)$, $\chi_2(x, \lambda)$ are two solutions of (5.1). Then $\chi_1(x, \lambda) - \chi_2(x, \lambda)$ is a solution of the problem (4.6a)–(4.6c). So, it is identically zero if λ is not an eigenvalue. Another proof of this assertion is included in the proof of the next theorem.

Theorem 5.1. *Suppose that λ is not an eigenvalue of (4.6a)–(4.6c). Let $\phi(\cdot, \lambda)$ satisfy the q-difference equation (5.1) and the boundary conditions (4.6b)–(4.6c), where $f(\cdot) \in L_q^2(0, a)$. Then*

$$\phi(x, \lambda) = \int_0^a G(x, t, \lambda)f(t)\, \mathrm{d}_q t, \qquad x \in \{0, aq^m, m \in \mathbb{N}\}, \tag{5.2}$$

where $G(x, t, \lambda)$ is Green's function of problem (4.6a)–(4.6c) and it is given by

$$G(x, t, \lambda) = \frac{-1}{\Delta(\lambda)}\begin{cases}\theta_2(x, \lambda)\theta_1(t, \lambda), & 0 \leqslant t \leqslant x,\\ \theta_1(x, \lambda)\theta_2(t, \lambda), & x < t \leqslant a.\end{cases} \tag{5.3}$$

Conversely the function $\phi(\cdot, \lambda)$ defined by (5.2) satisfies (5.1) and (4.6b)–(4.6c). Green's function $G(x, t, \lambda)$ is unique in the sense that if there exists another function $\widetilde{G}(x, t, \lambda)$ such that (5.2) is satisfied, then

$$G(x, t, \lambda) = \widetilde{G}(x, t, \lambda), \qquad \text{in} \quad L_q^2((0, a) \times (0, a)). \tag{5.4}$$

If $f(\cdot)$ is q-regular at zero, then (5.2) holds for all $x \in [0, a]$.

Proof. Using a q-analogue of the methods of variation of constants, a particular solution of the non-homogenous equation (5.1) may be given by

$$\phi(x,\lambda) = c_1(x)\theta_1(x,\lambda) + c_2(x)\theta_2(x,\lambda), \tag{5.5}$$

where $c_1(x)$, $c_2(x)$ are solutions of the first-order q-difference equations

$$D_{q,x}c_1(x) = \frac{q}{\Delta(\lambda)}\theta_2(qx,\lambda)f(qx), \qquad D_{q,x}c_2(x) = -\frac{q}{\Delta(\lambda)}\theta_1(qx,\lambda)f(qx). \tag{5.6}$$

From lemma 2.1, the functions $D_{q,x}c_i(x)$, $i = 1, 2$, are q-integrable on $[0, t]$ if and only if

$$\lim_{n\to\infty} tq^n\theta_i(tq^{n+1},\lambda)f(tq^{n+1}) = 0, \qquad i = 1, 2. \tag{5.7}$$

Define the q-geometric set A_f by

$$A_f := \{x \in [0,a]\colon \lim_{n\to\infty} xq^n|f(xq^n)|^2 = 0\}. \tag{5.8}$$

A_f is a q-geometric set containing $\{0, aq^m, m \in \mathbb{N}\}$ since $f \in L_q^2(0,a)$. Hence, $D_q c_i(\cdot)$, $i = 1, 2$, are q-integrable on $[0, x]$ for all $x \in A_f$ and appropriate solutions of (5.6) are given by

$$c_1(x) = c_1(0) + \frac{q}{\Delta(\lambda)}\int_0^x \theta_2(qt,\lambda)f(qt)\,\mathrm{d}_q t, \qquad x \in A_f \tag{5.9}$$

$$c_2(x) = c_2(a) + \frac{q}{\Delta(\lambda)}\int_x^a \theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t, \qquad x \in A_f. \tag{5.10}$$

That is the general solution of (5.1) is given by

$$\begin{aligned}\phi(x,\lambda) = c_1\theta_1(x,\lambda) + c_2\theta_2(x,\lambda) &+ \frac{q}{\Delta(\lambda)}\theta_1(x,\lambda)\int_0^x \theta_2(qt,\lambda)f(qt)\,\mathrm{d}_q t\\ &+ \frac{q}{\Delta(\lambda)}\theta_2(x,\lambda)\int_x^a \theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t,\end{aligned} \tag{5.11}$$

where $x \in A_f$, and c_1, c_2 are arbitrary constants. Now, we determine c_1, c_2 for which $\phi(x,\lambda)$ satisfies (4.6b), (4.6c). It easy to see that

$$\phi(0,\lambda) = c_1\theta_1(0,\lambda) + \left(c_2 + \frac{q}{\Delta(\lambda)}\int_0^a \theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t\right)\theta_2(0,\lambda),$$

$$\begin{aligned}D_{q^{-1}}\phi(0,\lambda) &= \lim_{\substack{n\to\infty\\ x\in A_f}} \frac{\phi(xq^n,\lambda) - \phi(0,\lambda)}{xq^n}\\ &= c_1 D_{q^{-1}}\theta_1(0,\lambda) + \left(c_2 + \frac{q}{\Delta(\lambda)}\int_0^a \theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t\right) D_{q^{-1}}\theta_2(0,\lambda).\end{aligned}$$

The boundary condition $a_{11}\phi(0,\lambda) + a_{12}D_{q^{-1}}\phi(0,\lambda) = 0$ implies that

$$\left(c_2 + \frac{q}{\Delta(\lambda)}\int_0^a \theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t\right) W_q(\theta_1,\theta_2)(0) = 0. \tag{5.12}$$

Therefore,

$$c_2 = \frac{-q}{\Delta(\lambda)}\int_0^a \theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t. \tag{5.13}$$

Hence,

$$\phi(x,\lambda) = c_1\theta_1(x,\lambda) + \frac{q}{\Delta(\lambda)}\int_0^x (\theta_1(x,\lambda)\theta_2(qt,\lambda) - \theta_2(x,\lambda)\theta_1(qt,\lambda))f(qt)\,\mathrm{d}_q t. \tag{5.14}$$

Now we compute $\phi(a,\lambda)$ and $D_{q^{-1}}\phi(a,\lambda)$. Indeed, from the definition of the q-integration (2.14) and relation (5.11)

$$\begin{aligned}\phi(a,\lambda) &= c_1\theta_1(a,\lambda) + \frac{q}{\Delta(\lambda)}\int_0^a (\theta_1(a,\lambda)\theta_2(qt,\lambda) - \theta_2(a,\lambda)\theta_1(qt,\lambda))f(qt)\,\mathrm{d}_q t\\ &= c_1\theta_1(a,\lambda) + \frac{q}{\Delta(\lambda)}\int_0^{q^{-1}a} (\theta_1(a,\lambda)\theta_2(qt,\lambda) - \theta_2(a,\lambda)\theta_1(qt,\lambda))f(qt)\,\mathrm{d}_q t\end{aligned}$$

and

$$\begin{aligned}D_{q^{-1}}\phi(a,\lambda) &= D_{q^{-1}}\theta_1(a,\lambda)\left(c_1 + \frac{q}{\Delta(\lambda)}\int_0^{q^{-1}a}\theta_2(qt,\lambda)f(qt)\,\mathrm{d}_q t\right)\\ &\quad - \frac{q}{\Delta(\lambda)}D_{q^{-1}}\theta_2(a,\lambda)\int_0^{q^{-1}a}\theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t.\end{aligned}$$

The boundary condition $a_{21}\phi_2(a,\lambda) + a_{22}D_{q^{-1}}\phi_2(a,\lambda) = 0$ implies

$$\left(c_1 + \frac{q}{\Delta(\lambda)}\int_0^{q^{-1}a}\theta_2(qt,\lambda)f(qt)\,\mathrm{d}_q t\right)W_q(\theta_1,\theta_2)(a) = 0. \tag{5.15}$$

Hence

$$c_1 = \frac{-q}{\Delta(\lambda)}\int_0^{q^{-1}a}\theta_2(qt,\lambda)f(qt)\,\mathrm{d}_q. \tag{5.16}$$

So for $x \in A_f$

$$\begin{aligned}\phi(x,\lambda) &= \frac{-q}{\Delta(\lambda)}\theta_2(x,\lambda)\int_0^x\theta_1(qt,\lambda)f(qt)\,\mathrm{d}_q t - \frac{q}{\Delta(\lambda)}\theta_1(x,\lambda)\int_x^{q^{-1}a}\theta_2(qt,\lambda)f(qt)\,\mathrm{d}_q t\\ &= \frac{-1}{\Delta(\lambda)}\theta_2(x,\lambda)\int_0^{qx}\theta_1(t,\lambda)f(t)\,\mathrm{d}_q t - \frac{1}{\Delta(\lambda)}\theta_1(x,\lambda)\int_{qx}^{a}\theta_2(t,\lambda)f(t)\,\mathrm{d}_q t\\ &= \frac{-1}{\Delta(\lambda)}\theta_2(x,\lambda)\int_0^{x}\theta_1(t,\lambda)f(t)\,\mathrm{d}_q t - \frac{1}{\Delta(\lambda)}\theta_1(x,\lambda)\int_{x}^{a}\theta_2(t,\lambda)f(t)\,\mathrm{d}_q t.\end{aligned}$$

proving (5.2), (5.3). Conversely, by direct computations, if $\phi(x,\lambda)$ is given by (5.2), then it is a solution of (5.1) and satisfies the boundary conditions (4.6b), (4.6c). To prove the uniqueness, suppose that there exists another function, $\widetilde{G}(x,t,\lambda)$, such that

$$\psi(x,\lambda) = \int_0^a \widetilde{G}(x,t,\lambda)f(t)\,\mathrm{d}_q t, \tag{5.17}$$

is a solution of (5.1) which satisfies (4.6b), (4.6c). For convenience, let

$$G(x,t,\lambda) = \begin{cases} G_1(x,t,\lambda), & 0 \leqslant t \leqslant x,\\ G_2(x,t,\lambda), & x \leqslant t \leqslant a,\end{cases}$$

$$\widetilde{G}(x,t,\lambda) = \begin{cases} \widetilde{G}_1(x,t,\lambda), & 0 \leqslant t \leqslant x,\\ \widetilde{G}_2(x,t,\lambda), & x \leqslant t \leqslant a.\end{cases}$$

By subtraction,

$$\int_0^a \{G(x,t,\lambda) - \widetilde{G}(x,t,\lambda)\}f(t)\,\mathrm{d}_q t = 0, \qquad x \in \{0, aq^m, m \in \mathbb{N}\}, \tag{5.18}$$

for all functions $f(t) \in L_q^2(0, a)$. Let us take $f(t) := \overline{G(x, t, \lambda) - \widetilde{G}(x, t, \lambda)}$ and $x = aq^m$, $m \in \mathbb{N}$. Then

$$\begin{aligned}\int_0^a |G(aq^m, t, \lambda) - \widetilde{G}(aq^m, t, \lambda)|^2 \, \mathrm{d}_q t &= \int_0^{aq^m} |G_1(aq^m, t, \lambda) - \widetilde{G_1}(aq^m, t, \lambda)|^2 \, \mathrm{d}_q t \\ &\quad + \int_{aq^m}^a |G_2(aq^m, t, \lambda) - \widetilde{G_2}(aq^m, t, \lambda)|^2 \, \mathrm{d}_q t \\ &= a(1-q) \sum_{n=0}^{\infty} q^n |G(aq^m, aq^n, \lambda) - \widetilde{G}(aq^m, aq^n, \lambda)|^2 = 0 \qquad (5.19)\end{aligned}$$

Therefore, from (5.19) we conclude that

$$G(aq^m, aq^n, \lambda) = \widetilde{G}(aq^m, aq^n, \lambda), \qquad m, n \in \mathbb{N}.$$

If $f(\cdot)$ is q-regular at zero, then $A_f \equiv [0, a]$ and (5.2) will be defined for all $x \in [0, a]$. □

Theorem 5.2. *Green's function has the following properties:*

(i) $G(x, t, \lambda)$ *is continuous at the point* $(0, 0)$.
(ii) $G(x, t, \lambda) = G(t, x, \lambda)$.
(iii) For each fixed $t \in (0, qa]$, *as a function of* x, $G(x, t, \lambda)$ *satisfies the q-difference equation (4.6a) in the intervals* $[0, t)$, $(t, a]$ *and it also satisfies the boundary conditions (4.6b), (4.6c).*
(iv) Let λ_0 *be a zero of* $\Delta(\lambda)$. *Then* λ_0 *can be a simple pole of the function* $G(x, t, \lambda)$, *and in this case*

$$G(x, t, \lambda) = \frac{-\psi_0(x)\psi_0(t)}{\lambda - \lambda_0} + \widetilde{G}(x, t, \lambda), \qquad (5.20)$$

where $\widetilde{G}(x, t, \lambda)$ *is an analytic function of* λ *in a neighbourhood of* λ_0 *and* $\psi_0(\cdot)$ *is a normalized eigenfunction corresponding to* λ_0.

Proof. (i) Follows from the continuity of $\theta_1(\cdot, \lambda), \theta_2(\cdot, \lambda)$ at zero for each fixed $\lambda \in \mathbb{C}$ and (ii) is easy to be checked. Now, we prove (iii). Let $t \in (0, qa]$ be fixed. If $x \in [0, t]$, then

$$G(x, t, \lambda) = \frac{1}{\Delta(\lambda)} \theta_1(x, \lambda)\theta_2(t, \lambda).$$

So,

$$\ell G(x, t, \lambda) = \frac{1}{\Delta(\lambda)} \theta_2(t, \lambda) \ell \theta_1(x, \lambda) = \frac{\lambda}{\Delta(\lambda)} \theta_2(t, \lambda)\theta_1(x, \lambda) = \lambda G(x, t, \lambda).$$

Similarly if $x \in [t, a]$. From (4.20) and (5.3), we have

$$a_{11} G(0, t, \lambda) + a_{12} D_{q^{-1}} G(0, t, \lambda) = \frac{1}{\Delta(\lambda)} \theta_2(t, \lambda)\{a_{11}\theta_1(0, \lambda) + a_{12} D_{q^{-1}} \theta_1(0, \lambda)\} = 0,$$

$$a_{21} G(a, t, \lambda) + a_{22} D_{q^{-1}} G(a, t, \lambda) = \frac{1}{\Delta(\lambda)} \theta_1(t, \lambda)\{a_{21}\theta_1(a, \lambda) + a_{22} D_{q^{-1}} \theta_1(a, \lambda)\} = 0.$$

(iv) Let λ_0 be a pole of $G(x, t, \lambda)$, and $R(x, t)$ be the residue of $G(x, t, \lambda)$ at $\lambda = \lambda_0$. From (4.22) and (4.24), we obtain

$$\begin{aligned} R(x, t) &= \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0) G(x, t, \lambda) = k_0^{-1} \theta_1(x, \lambda_0)\theta_1(t, \lambda_0) \lim_{\lambda \to \lambda_0} \frac{\lambda - \lambda_0}{\Delta(\lambda)} \\ &= -\frac{\theta_1(x, \lambda_0)\theta_1(t, \lambda_0)}{\int_0^a |\theta_1(u, \lambda)|^2 \, \mathrm{d}_q u} = -\psi_0(x, \lambda_0)\psi_1(t, \lambda_0). \end{aligned}$$

□

6. Eigenfunctions expansion formula

In this section, the existence of a countable sequence of eigenvalues of ℓ with no finite limit points will be proved by using the spectral theorem of compact self-adjoint operators in Hilbert spaces (see, e.g., [6]). Moreover it will be proved that the corresponding eigenfunctions form an orthonormal basis of $L_q^2(0,a)$. We define the operator $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \to L_q^2(0,a)$ to be $\mathcal{L}y = \ell y$ for all $y \in \mathcal{D}_{\mathcal{L}}$, where $\mathcal{D}_{\mathcal{L}}$ is the subspace of $L_q^2(0,a)$ consisting of those complex-valued functions y that satisfies (4.6b), (4.6c) such that $D_q y(\cdot)$ is q-regular at zero and $D_q^2 y(\cdot)$ lies in $L_q^2(0,a)$. Thus $\mathcal{L}$ is the difference operator generated by the difference expression ℓ and the boundary conditions (4.6b), (4.6c). By $\mathcal{L}(y) = \lambda y$, we mean that $\ell(y) = \lambda y$ and y satisfies (4.6b), (4.6c). The operator $\mathcal{L}$ has the same eigenvalues of the basic Sturm–Liouville problem (4.6a)–(4.6c). We assume without any loss of generality that $\lambda = 0$ is not an eigenvalue. Thus $\ker \mathcal{L} = \{0\}$. From the previous section the solution of the problem

$$(\mathcal{L}y)(x) = f(x), \qquad f \in L_q^2(0,a), \tag{6.1}$$

is given uniquely in $L_q^2(0,a)$ by

$$y(x) = \int_0^a G(x,t) f(t)\, \mathrm{d}_q t, \tag{6.2}$$

where

$$G(x,t) = G(x,t,0) = \begin{cases} c\theta_1(t)\theta_2(x), & 0 \leqslant t \leqslant x \\ c\theta_1(x)\theta_2(t), & x \leqslant t \leqslant a, \end{cases} \qquad c := -\frac{1}{W_q(\theta_1,\theta_2)}. \tag{6.3}$$

Replacing $f(\cdot)$ by $\lambda y(\cdot)$ in (6.1), then the eigenvalue problem

$$(\mathcal{L}y)(x) = \lambda y(x), \tag{6.4}$$

is equivalent to the following basic Fredholm integral equation of the second kind:

$$y(x) = \lambda \int_0^a G(x,t) y(t)\, \mathrm{d}_q t, \qquad \text{in} \quad L_q^2(0,a). \tag{6.5}$$

Let $\mathcal{G}$ be the integral operator

$$\mathcal{G} : L_q^2(0,a) \to L_q^2(0,a), \qquad (\mathcal{G}f)(x) = \int_0^a G(x,t) f(t)\, \mathrm{d}_q t. \tag{6.6}$$

We prove that

$$(\mathcal{L}\mathcal{G})f = f, \qquad f \in L_q^2(0,a). \tag{6.7}$$

We show first that $y = \mathcal{G}f \in \mathcal{D}_{\mathcal{L}}$. From (6.5) and (6.6)

$$y(x) = (\mathcal{G}f)(x) = \theta_2(x) y_1(x) + \theta_1(x) y_2(x), \tag{6.8}$$

where

$$y_1(x) = c\int_0^x \theta_1(t) f(t)\, \mathrm{d}_q t, \qquad y_2(x) = c\int_x^a \theta_2(t) f(t)\, \mathrm{d}_q t.$$

Thus, for all $x \in A_f$, cf. (5.8),

$$D_q y(x) = D_q\theta_2(x) y_1(qx) + D_q\theta_1(x) y_2(qx), \tag{6.9}$$

$$D_q^2 y(x) = -q v(qx)(y)(qx) - q f(qx) \in L_q^2(0,a). \tag{6.10}$$

Since $D_q\theta_i(x,\lambda)$, $y_i(x)$, $i = 1, 2$, are q-regular at zero, then so is $D_q y(\cdot)$ and

$$D_{q^{-1}} y(0) = D_q y(0) = \lim_{\substack{n\to\infty \\ x\in A_f}} \frac{y(xq^n) - y(0)}{xq^n} = D_{q^{-1}}\theta_2(0) y_2(0),$$

$y_1(0) = 0$, and $y_2(a) = 0$, then

$$a_{11}y(0) + a_{12}D_{q^{-1}}y(0) = (a_{11}\theta_1(0) + a_{12}D_{q^{-1}}\theta_1(0))y_2(0) = 0,$$

and

$$a_{21}y(a) + a_{22}D_{q^{-1}}y(a) = (a_{21}\theta_2(a) + a_{22}D_{q^{-1}}\theta_2(a))y_1(a) = 0.$$

Thus $y \in \mathcal{D}_{\mathcal{L}}$. It follows from (6.10) that $\mathcal{L}y = (\mathcal{L}\mathcal{G})(f) = f$. Hence we have established (6.7). Also, we can see that

$$(\mathcal{G}\mathcal{L})(y) = y, \qquad y \in \mathcal{D}_{\mathcal{L}}. \tag{6.11}$$

Indeed, replacing f in (6.7) by $\mathcal{L}y$, we get $\mathcal{L}y = \mathcal{L}\mathcal{G}\mathcal{L}y$. Thus $y = \mathcal{G}\mathcal{L}y$ since $\mathcal{L}$ is assumed to be injective. It follows from (6.7) and (6.11) that $\ker \mathcal{G} = \{0\}$ and ϕ is an eigenfunction of $\mathcal{G}$ with an eigenvalue μ if and only if ϕ is an eigenfunction of $\mathcal{L}$ with an eigenvalue $1/\mu$.

Theorem 6.1. *The operator $\mathcal{G}$ is compact and self-adjoint.*

Proof. Let $f, h \in L_q^2(0, a)$. Since $G(x, t)$ is a real-valued function defined on $[0, a] \times [0, a]$ and $G(x, t) = G(t, x)$, then for $f, h \in L_q^2(0, a)$,

$$\begin{aligned}\langle \mathcal{G}(f), h\rangle &= \int_0^a (\mathcal{G}f)(x)\overline{h(x)}\,\mathrm{d}_q x = \int_0^a \int_0^a G(x, t)f(t)\overline{h(x)}\,\mathrm{d}_q t\,\mathrm{d}_q x \\ &= \int_0^a f(t)\overline{\left(\int_0^a G(t, x)h(x)\,\mathrm{d}_q x\right)}\,\mathrm{d}_q t = \langle f, \mathcal{G}(h)\rangle,\end{aligned}$$

i.e. $\mathcal{G}$ is self-adjoint. Let $\{\phi_{ij}(x, t) = \phi_i(x)\phi_j(t)\}$ be an orthonormal basis of $L_q^2((0, a) \times (0, a))$. Consequently $G = \sum_{i,j=1}^{\infty}\langle G, \phi_{ij}\rangle\phi_{ij}$. For $n \in \mathbb{Z}^+$ set $G_n = \sum_{i,j=1}^{n}\langle G, \phi_{ij}\rangle\phi_{ij}$, and let $\mathcal{G}_n$ be the finite rank integral operator defined on $L_q^2(0, a)$ by

$$\mathcal{G}_n(f)(x) := \int_0^a G_n(x, t)f(t)\,\mathrm{d}_q t, \qquad \text{in} \quad L_q^2(0, a). \tag{6.12}$$

Obviously $\mathcal{G}_n$ is compact for all $n \in \mathbb{N}$. From Cauchy–Schwarz' inequality

$$\begin{aligned}\|(\mathcal{G} - \mathcal{G}_n)(f)\| &= \left(\int_0^a |(\mathcal{G} - \mathcal{G}_n)(f)(x)|^2\,\mathrm{d}_q t\right)^{1/2} \\ &= \left(\int_0^a \left|\int_0^a (G - G_n)(x, t)(f)(t)\,\mathrm{d}_q t\right|^2\,\mathrm{d}_q x\right)^{1/2} \\ &\leqslant \left(\int_0^a \int_0^a |(G - G_n)(x, t)|^2\,\mathrm{d}_q t\,\mathrm{d}_q x\right)^{1/2}\left(\int_0^a |f(x)|^2\,\mathrm{d}_q x\right)^{1/2} \\ &= \|G - G_n\|_2\|f\|,\end{aligned}$$

then

$$\|\mathcal{G} - \mathcal{G}_n\| \leqslant \|G - G_n\|_2 \to 0 \qquad \text{as} \quad n \to \infty.$$

This completes the proof. □

Corollary 6.2. *The eigenvalues of the operator $\mathcal{L}$ form an infinite sequence $\{\lambda_k\}_{k=1}^{\infty}$ of real numbers which can be ordered so that*

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n| < \cdots \to \infty \qquad \textit{as} \quad n \to \infty.$$

The set of all normalized eigenfunctions of $\mathcal{L}$ forms an orthonormal basis for $L_q^2(0, a)$.

Proof. Since $\mathcal{G}$ is a compact self-adjoint operator on $L_q^2(0,a)$, then $\mathcal{G}$ has an infinite sequence of non-zero real eigenvalues $\{\mu_n\}_{n=1}^{\infty}, \subseteq R, \mu_n \to 0$ as $n \to \infty$. Let $\{\phi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\mu_n\}_{n=1}^{\infty}$. From the spectral theorem of compact self-adjoint operators, we have,

$$\mathcal{G}(f) = \sum_{n=0}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n. \tag{6.13}$$

Since the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the operator $\mathcal{L}$ are the reciprocal of those of $\mathcal{G}$, then

$$|\lambda_n| = \frac{1}{|\mu_n|} \to \infty \qquad \text{as} \quad n \to \infty. \tag{6.14}$$

Let $y \in \mathcal{D}_{\mathcal{L}}$. Then, $y = G(f)$, for some $f \in L_q^2(0,a)$. Consequently,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n = \sum_{n=0}^{\infty} \mu_n \langle ly, \phi_n \rangle \phi_n \\ &= \sum_{n=0}^{\infty} \mu_n \langle y, \ell\phi_n \rangle \phi_n = \sum_{n=0}^{\infty} \langle y, \phi_n \rangle \phi_n. \end{aligned}$$

If zero is an eigenvalue of $\mathcal{L}$. Then, we can choose $r \in \mathbb{R}$ such that r is not an eigenvalue of $\mathcal{L}$. Now, applying the above result on $\mathcal{L} - rI$ in place of $\mathcal{L}$ yields the corollary. □

Example 1. Consider the q-Sturm–Liouville boundary value problem

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) = \lambda y(x), \tag{6.15}$$

with the q-Dirichlet conditions

$$U_1(y) = y(0) = 0, \qquad U_2(y) = y(1) = 0. \tag{6.16}$$

A fundamental set of solutions of (6.15) is

$$\phi_1(x,\lambda) = \cos(\sqrt{\lambda}x; q), \qquad \phi_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}x; q)}{\sqrt{\lambda}}. \tag{6.17}$$

Now, the eigenvalues of problem (6.15) are the zeros of the determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(\phi_1) & U_2(\phi_1) \\ U_1(\phi_2) & U_2(\phi_2) \end{vmatrix} = \phi_2(1,\lambda) = \frac{\sin(\sqrt{\lambda}; q)}{\sqrt{\lambda}}. \tag{6.18}$$

Hence, the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are the zeros of $\sin(\sqrt{\lambda}; q)$. From (2.11),

$$\lambda_n = (1-q)^{-2} q^{-2n+2\epsilon_n^{(-1/2)}}, \qquad n \geqslant 1, \tag{6.19}$$

for sufficiently large n and the corresponding set of eigenfunctions $\left\{\frac{\sin(\sqrt{\lambda_n}; q)}{\sqrt{\lambda_n}}\right\}_{n=1}^{\infty}$ is an orthogonal basis of $L_q^2(0,1)$. In the previous notation

$$\theta_1(x,\lambda) = \frac{\sin(\sqrt{\lambda}x; q)}{\sqrt{\lambda}}, \tag{6.20}$$

and

$$\theta_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}; q)}{\sqrt{\lambda}} \cos(\sqrt{\lambda}x; q) + \cos(\sqrt{\lambda}; q) \frac{\sin(\sqrt{\lambda}x; q)}{\sqrt{\lambda}}. \tag{6.21}$$

So, if λ is not an eigenvalue, Green's function is given by

$$G(x,t,\lambda)=\frac{\sin(\sqrt{\lambda}t;q)}{\sin(\sqrt{\lambda};q)}\left(\cos(\sqrt{\lambda}x;q)\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}-\cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}\right), \tag{6.22}$$

for $0\leqslant t\leqslant x$, and

$$G(x,t,\lambda)=\frac{\sin(\sqrt{\lambda}x;q)}{\sin(\sqrt{\lambda};q)}\left(\cos(\sqrt{\lambda}t;q)\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}-\cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}t;q)}{\sqrt{\lambda}}\right), \tag{6.23}$$

for $x\leqslant t\leqslant 1$. Since $\lambda=0$ is not an eigenvalue, then Green function $G(x,t)$ is nothing but

$$G(x,t)=G(x,t,0)=\begin{cases}t(1-x), & 0\leqslant t\leqslant x,\\ x(1-t), & x\leqslant t\leqslant 1.\end{cases}$$

Hence the boundary value problem (6.15), (6.16) is equivalent to the basic Fredholm integral equation

$$y(x)=\lambda\int_0^1 G(x,t)y(t)\,\mathrm{d}_qt. \tag{6.24}$$

Example 2. Consider equation (6.15) with the q-Neumann boundary conditions

$$U_1(y)=D_{q^{-1}}y(0)=0, \qquad U_2(y)=D_{q^{-1}}y(1)=0. \tag{6.25}$$

In this case $\theta_1(x,\lambda)=\cos(\sqrt{\lambda}x;q)$, and

$$\theta_2(x,\lambda)=\cos(\sqrt{\lambda}q^{-1/2};q)\cos(\sqrt{\lambda}x;q)+\sqrt{q}\sin(\sqrt{\lambda}q^{-1/2};q)\sin(\sqrt{\lambda}x;q).$$

Since $\Delta(\lambda)=\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)$. Then $\lambda_0=0$ and for sufficiently large n, the eigenvalue are given by

$$\lambda_n=q^{-2n+1+2\epsilon_n^{(-1/2)}}, \qquad n\geqslant 1 \tag{6.26}$$

Therefore, $\{1,\cos(\sqrt{\lambda_n}x;q)\}_{n=1}^{\infty}$ is an orthogonal basis of $L_q^2(0,1)$. If λ is not an eigenvalue, then Green's function $G(x,t,\lambda)$ is defined for $x,t\in[0,a]\times[0,a]$ by

$$\begin{aligned}G(x,t,\lambda)=-&\frac{\cos(\sqrt{\lambda}t;q)}{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)}(\cos(\sqrt{\lambda}q^{-1/2};q)\cos(\sqrt{\lambda}x;q)\\&+\sqrt{q}\sin(\sqrt{\lambda}q^{-1/2};q)\sin(\sqrt{\lambda}x;q)),\end{aligned}$$

and

$$\begin{aligned}G(x,t,\lambda)=-&\frac{\cos(\sqrt{\lambda}x;q)}{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)}(\cos(\sqrt{\lambda}q^{-1/2};q)\cos(\sqrt{\lambda}t;q)\\&+\sqrt{q}\sin(\sqrt{\lambda}q^{-1/2};q)\sin(\sqrt{\lambda}t;q)), \qquad x\leqslant t\leqslant 1.\end{aligned}$$

The operator $\mathcal{L}$ associated with problem (6.15), (6.25) is not invertible since zero is an eigenvalue.

Example 3. Consider (6.15) with the following boundary conditions:

$$U_1(y)=y(0)=0, \qquad U_2(y)=y(1)+D_{q^{-1}}y(1)=0. \tag{6.27}$$

Then $\Delta(\lambda)=\phi_2(1,\lambda)+D_{q^{-1}}\phi_2(1,\lambda)=\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}+\cos(\sqrt{\lambda}q^{-1/2};q)$. The eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of this boundary value problem are the solutions of the equation

$$\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}=-\cos(\sqrt{\lambda}q^{-1/2};q) \tag{6.28}$$

and the corresponding eigenfunctions are $\left\{\frac{\sin(\sqrt{\lambda_n};q)}{\sqrt{\lambda_n}}\right\}_{n=1}^{\infty}$. The functions $\theta_1(x,\lambda)$ and $\theta_2(x,\lambda)$ are

$$\theta_1(x,\lambda) = \frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}},$$

$$\theta_2(x,\lambda) = \left(\cos(\sqrt{\lambda}q^{-1/2};q) + \frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}\right)\cos(\sqrt{\lambda x;q})$$
$$- (-\sqrt{\lambda q}\sin(\sqrt{\lambda}q^{-1/2};q) + \cos(\sqrt{\lambda};q))\frac{\sin(\sqrt{\lambda x;q})}{\sqrt{\lambda}}.$$

If λ is not an eigenvalue, then Green's function $G(x,t,\lambda)$ is defined to be

$$G(x,t,\lambda) = \frac{-1}{\sin(\sqrt{\lambda};q) + \sqrt{\lambda}\cos(\sqrt{\lambda}q^{-1/2};q)}\begin{cases}\sin(\sqrt{\lambda}t;q)\theta_2(x,\lambda), & 0 \leqslant t \leqslant x,\\ \sin(\sqrt{\lambda}x;q)\theta_2(t,\lambda), & x \leqslant t \leqslant 1.\end{cases}$$

and

$$G(x,t) = \frac{-1}{2}\begin{cases} t(2-x), & 0 \leqslant t \leqslant x,\\ x(2-t), & x \leqslant t \leqslant 1.\end{cases}$$

Therefore the boundary value problem (6.15), (6.27) is equivalent to the basic Fredholm integral equation

$$y(x) = \lambda \int_0^1 G(x,t)y(t)\,\mathrm{d}_q t.$$

Remark 1. Let $r(\cdot)$ be a real-valued function defined on $[0, q^{-1}a]$ such that $r(x) \neq 0$ for all $x \in [0, q^{-1}a]$ and $D_{q^{-1}}r(0)$ exists. Let $w(\cdot)$ be a real-valued function defined on $[0,a]$ and positive on $\{0, aq^n, n \in \mathbb{N}\}$. The Sturm–Liouville problem (4.6*a*)–(4.6*c*) may be defined for $0 \leqslant x \leqslant a < \infty$, $\lambda \in \mathbb{C}$ to be

$$M(y) := \frac{-1}{q}D_{q^{-1}}(r(x)D_q y(x)) + \nu(x)y(x) = \lambda w(x)y(x), \tag{6.29a}$$

$$U_1(y) := a_{11}y(0) + a_{12}(rD_q y)(0) = 0, \tag{6.29b}$$

$$U_2(y) := a_{21}y(a) + a_{22}(rD_q y)(aq^{-1}) = 0, \tag{6.29c}$$

where the functions $\nu(\cdot)$ and the constants $\{a_{ij}\}$, $i, j \in \{1,2\}$, are as in section 4. In this case we will have the Lagrange identity

$$\int_0^a ((My)(x)\overline{z(x)} - y(x)\overline{(Mz)(x)})\,\mathrm{d}_q x = [y,\bar{z}](a) - \lim_{n\to\infty}[y,\bar{z}](aq^n), \tag{6.30}$$

where

$$[y,z](x) = y(x)(rD_q z)(xq^{-1}) - (rD_q y)(xq^{-1})z(x). \tag{6.31}$$

If $y(\cdot)$, $z(\cdot)$ and $r(\cdot)$ are q-regular at zero, then Lagrange's identity (6.30) will be nothing but

$$\int_0^a ((My)(x)\overline{z(x)} - y(x)\overline{(Mz)(x)})\,\mathrm{d}_q x = [y,\bar{z}](a) - [y,\bar{z}](0). \tag{6.32}$$

Problem (6.29*a*)–(6.29*c*) is formally self-adjoint in $L_q^2([0,a]; w(\cdot))$, where $L_q^2([0,a]; w(\cdot))$ is the Hilbert space

$$L_q^2([0,a]; w(\cdot)) := \{f : [0,a] \to \mathbb{C} : \int_0^a |f(x)|^2 w(x)\,\mathrm{d}_q x < \infty\} \tag{6.33}$$

with the inner product

$$\langle f, g\rangle_w := \int_0^a f(x)\overline{g(x)}w(x)\,\mathrm{d}_q x, \qquad f, g \in L_q^2((0,a); w(\cdot)). \tag{6.34}$$

In a way similar to the theory developed in section 5, the eigenvalue problem (6.29*a*)–(6.29*c*) is equivalent to the q-type Fredholm integral equation

$$y(x) = \lambda \int_0^a G(x,\xi)y(\xi)w(\xi)\,\mathrm{d}_q\xi, \tag{6.35}$$

where $G(x,\xi)$ is Green's function associated with the problem. Consequently all results established above hold in this setting. In particular, problem (6.29*a*)–(6.29*c*) has a countable set of real simple eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$ with no finite limit points. A corresponding set of eigenfunctions $\{\phi_k(\cdot)\}_{k=0}^{\infty}$ is an orthonormal basis of $L_q^2((0,a); w(\cdot))$.

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